98-18

Note on the tail of the Overflow-time distribution

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Assuming a basic knowledge of queueing theory, we will look at the tail of Overflow-time distribution. A recent observation suggested that for certain queueing systems, the distribution of the "Overflow-time" has an exponential tail. One example of such a system is the $M(\lambda)/M(\mu)/1/b$ queue.

The $M(\lambda)/M(\mu)/1/b$ queueing system has a single server attending traffic with a Poisson arrivals rate λ, service takes place with an exponential service rate μ and the buffer is of size b. We can model the queue-length process using a finite state markov chain with state space $S = \{0, ..., b\}$ and transition rates;

$$\begin{aligned} P_{i,i+1} &= \lambda \quad for \quad 0 \le i \le b-1 \\ P_{i,i-1} &= \mu \quad for \quad 1 \le i \le b \end{aligned}$$

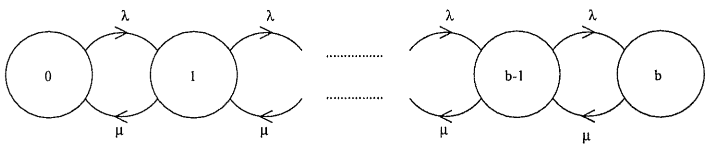

Figure 1: Markov Chain with state space S

The overflow-time, T, of the queue is the time taken for the non-empty queue to fill up its buffer without becoming non-empty in the process. Let Q_t denote the length of the queue at time t. If there exist $S, T \in \mathbb{R}$ such that

- $Q_S = 0$
- $Q_{S+j} \in \{1, ..., b-1\}, \forall j$ with $0 < j < T$
- $Q_{S+T} = b$

then we call T an overflowtime.

This translates into the time it spends in the states $S' = \{1, ..., b-1\}$ after leaving state 0 and before entering state b.

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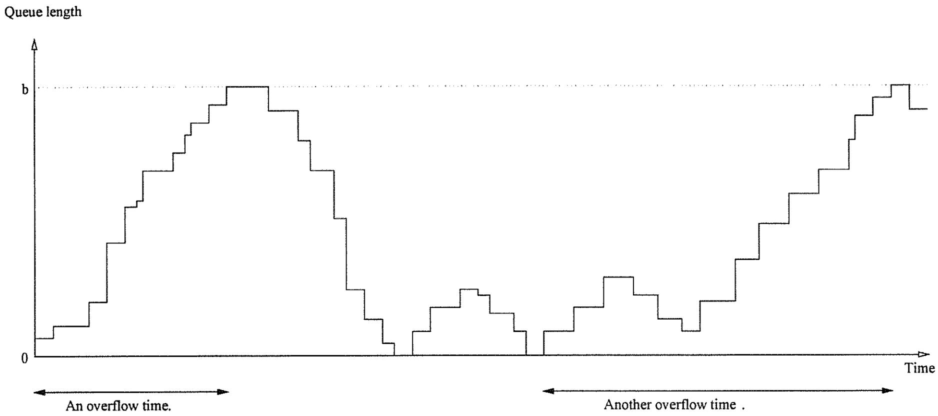

Figure 2: The queue length process showing some overflow times.

Conjecture 1 *For large t, $P(T > t) \approx e^{xt}$ where x is the largest eigenvalue of the matrix P'.*

What we are interested in is the tail of the fill-time(overflow-time) distribution so we must look at the asymptotic decay rate of the probability that the queue length remains in S'.

The decay rate is given by the largest eigenvalue of transition rate matrix $P' := P_{i+1,j+1}$ with $0 \leq i, j \leq b-2$ where

$$P' = \begin{pmatrix} -\lambda-\mu & \lambda & 0 & \cdots & \cdots & \cdots & \cdots \\ \mu & -\lambda-\mu & \lambda & \cdots & \cdots & \cdots & \cdots \\ 0 & \mu & -\lambda-\mu & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \mu & -\lambda-\mu & \lambda & 0 \\ \cdots & \cdots & \cdots & \cdots & \mu & -\lambda-\mu & \lambda \\ \cdots & \cdots & \cdots & \cdots & \cdots & \mu & -\lambda-\mu \end{pmatrix}.$$

This is a $(b-1)*(b-1)$ tri-diagonal Toeplitz matrix and is the transition matrix of the $M(\lambda)/M(\mu)/1/b$ queueing system with the states $\{0, b\}$ discarded. The eigenvalues are given by [1] as

$$x_k = -2\sqrt{\lambda\mu}\cos\frac{k\pi}{b-1} - \lambda - \mu \quad \text{for } 1 \leq k \leq b-2,$$

the largest of which will be

$$\begin{aligned} x_{max} &:= x_{b-2} \\ &= 2\sqrt{\lambda\mu}\cos\frac{\pi}{b-1} - \lambda - \mu. \end{aligned}$$

By using an electrical computer, we simulated the $M(\lambda)/M(\mu)/1/b$ queue and made a plot of the $\log P(T > t)$ where t is indexed by the horizontal coordinate of the graphs. We can see that the asymptotic decay rate given by the largest eigenvalue is in proximity with the results from the simulated queue.

An interesting property of x_{max}, as we have derived it, is that as the buffer size b tends towards infinity, $x_{max} \to 2\sqrt{\lambda\mu} - \lambda - \mu$.

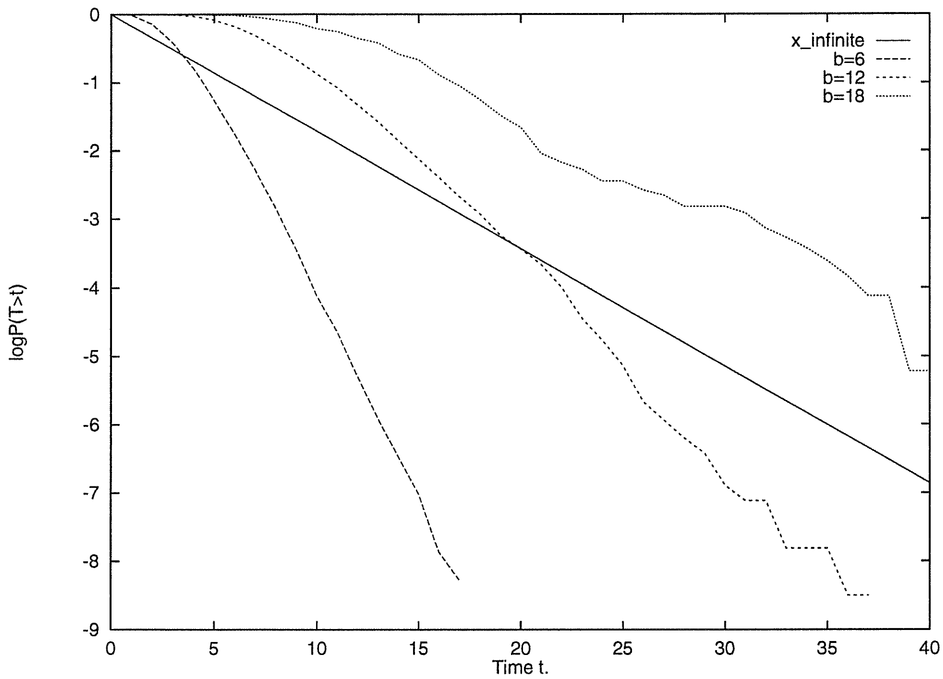

Figure 3: As the buffer size increases, the asymptotic slope approaches $2\sqrt{\lambda\mu} - \lambda - \mu$.

References

[1] R.B. Marr and G.H. Vineyard (1988) Five-diagonal Toeplitz determinants and their relation to Chebyshev polynomials. *SIAM J. Matrix Anal. Appl.* **9** 579–586.

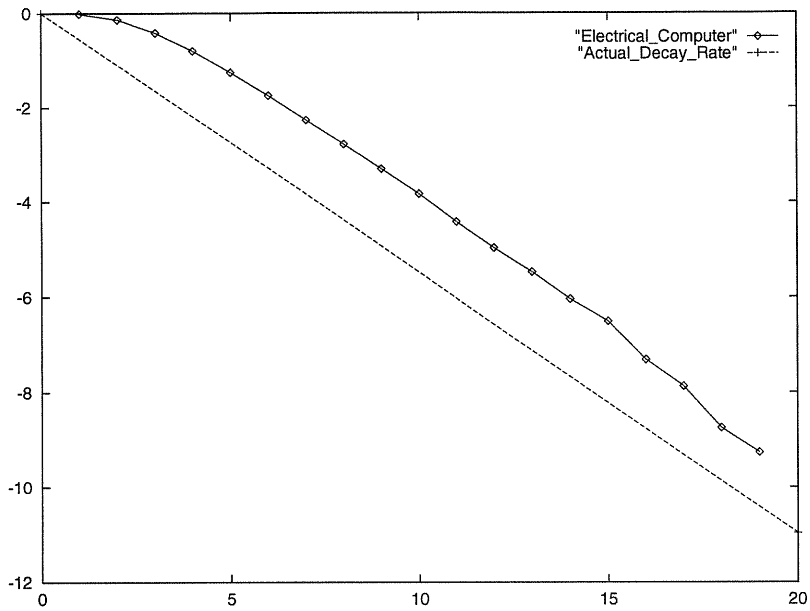

Figure 4: Simulation run with $\lambda = 2.0, \mu = 1.0$ and $b = 6$ for a length of 4,000,000 cycles.

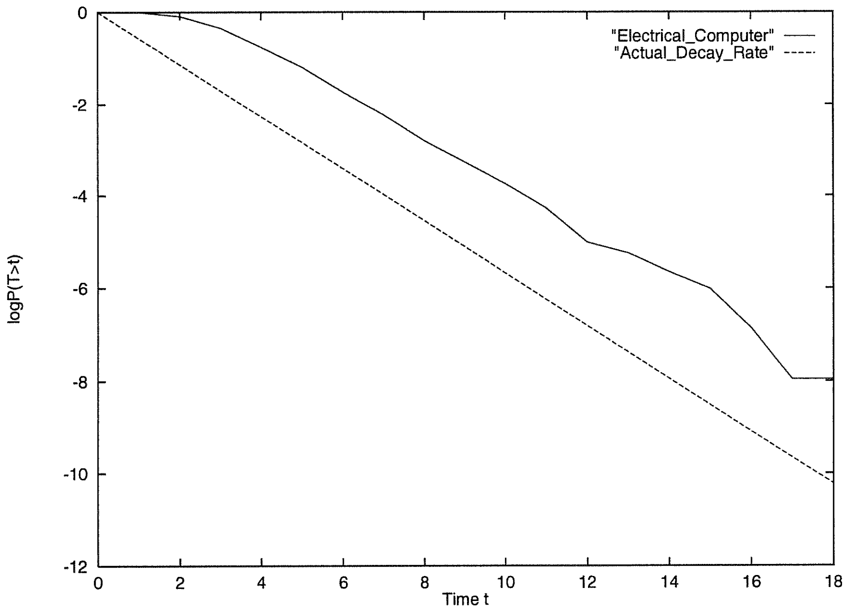

Figure 5: Simulation run with $\lambda = 5.0, \mu = 3.0$ and $b = 12$ for a length of 5,000,000 cycles.